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On the escape of particles from cosmic ray modified shocks

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ABSTRACT

The solution of the problem of particle acceleration in the non-linear regime, when the dynamical reaction of the accelerated particles cannot be neglected, shows strong shock modification. When stationarity is imposed by hand, the solution may show a prominent energy flux away from the shock towards upstream infinity. This feature is peculiar of cosmic ray modified shocks, while being energetically insignificant in the test particle regime. The escape flux appears also in situations in which it is physically impossible to have particle escape towards upstream infinity, thereby leading to question its interpretation. We show here that the appearance of an escape flux is due to the unphysical assumption of stationarity of the problem, and in a realistic situation it translates to an increase of the value of the maximum momentum when the shock velocity is constant. On the other hand, when the shock velocity decreases (for instance during the Sedov-Taylor phase of a supernova explosion), escape to upstream infinity is possible for particles with momenta in a narrow range close to the maximum momentum.

Key words: acceleration of particles - shock waves

1 INTRODUCTION

Kinetic approaches to non-linear particle acceleration (Malkov (1997); Malkov & O'C Drury (2001); Blasi (2002, 2004); Amato & Blasi (2005)) allow us to calculate the spectrum and the spatial distribution (including the absolute normalization) of the particles accelerated at the shock front, even

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in the case when the diffusion coefficient is the result of magnetic field amplification by streaming instability induced by the accelerated particles themselves (Amato & Blasi (2006)). These approaches, as well as others that have appeared in the literature (for instance Berezhko & Ellison (1999)), are based on the assumption of stationarity of the acceleration process. In all these cases the calculations show that the shock is strongly modified by the presence of cosmic rays, and that the spectra are concave, with a slope at momenta close to the maximum momentum p_{max} which is flatter than p^{-4}. All these calculations, independently of the techniques used to solve the equations, predict an escape flux of particles (and energy) towards upstream infinity: the shock becomes radiative, which is one of the very reasons why the shock modification becomes effective (namely the total compression factor becomes larger than 4).

It is worth recalling that the assumption of stationarity was widely used also in the context of two-fluid models (Drury & Vöelk (1981a,b)) but the appearance of an escape flux apparently was not recognized.

Here we discuss the physical meaning of this escape flux, with a special attention for the role it plays during the different phases in the expansion of a shell supernova remnant. A stationary solution of the transport equation without energy losses or escape cannot exist in the test-particle regime, nor in the non-linear one, although in the first case a quasi-stationary solution can be found for $p \ll p_{max}$. The non-stationarity reflects into an increase of p_{max} with time if the shock velocity remains constant (free expansion phase). When the shock starts slowing down (Sedov-Taylor phase), the maximum momentum either increases very slowly or decreases with time. In the latter case, particles with momenta larger than the current p_{max} can leave the shock region carrying energy toward upstream infinity.

On the other hand, the theoretical prediction of an escape flux during the free expansion phase is clearly unphysical, and should be considered as a warning that the stationary (or quasi-stationary) solutions are inadequate to describe this phase. If the stationary approach were used, nonetheless, one would still predict an escape flux and a strong shock modification. However this prediction would not be strictly correct, but rather signal for the need of treating the problem in a time dependent way.

We discuss at length the phenomenological implications of the escape of particles from a supernova shell in the Sedov-Talor phase, especially for the origin of cosmic rays.

The paper is organized as follows: in §2 we discuss the implications of the assumption of stationarity of the acceleration process. In §3 we discuss the escape flux based on the most general version of the conservation equations. In §4 we discuss how the escape flux is connected to the existence of a maximum momentum in the distribution of accelerated particles. In §5 we apply our calculations to the different stages of evolution of a supernova remnant. We conclude in §6.

2 THE ASSUMPTION OF STATIONARITY

The standard solution of the stationary transport equation

$$u(x)\frac{\partial f(x,p)}{\partial x} = \frac{\partial}{\partial x}\left[D(p)\frac{\partial f(x,p)}{\partial x}\right] + \frac{1}{3}\frac{du}{dx}\frac{\partial f}{\partial p} + Q \tag{1}$$

leads, in the test-particle regime, to the well known power-law spectrum of accelerated particles $f(p) \propto p^{-\alpha}$, with $\alpha = 3r/(r-1)$ where r is the compression factor at the shock.

The power law extends to infinitely large momenta. Since for ordinary non-relativistic gaseous shocks $r < 4$ (namely $\alpha > 4$), the total energy in the form of accelerated particles remains finite. This solution is found by imposing as boundary condition at upstream infinity ($x = -\infty$) that $f(-\infty) = 0$ and $\partial f(-\infty)/\partial x = 0$.

If the boundary condition $f(x = x_0) = 0$ is used, instead, at some finite distance $x_0 < 0$ upstream, the solution of the transport equation is easily calculated to be

$$f(x,p) = \frac{f_0(p)}{1-\exp\left(\frac{u_1 x_0}{D(p)}\right)}\left[\exp\left(\frac{u_1 x}{D(p)}\right) - \exp\left(\frac{u_1 x_0}{D(p)}\right)\right], \tag{2}$$

where

$$f_0(p) = K \exp\left\{-\frac{3u_1}{u_1-u_2}\int_{p_{inj}}^{p}\frac{dp'}{p'}\frac{1}{1-\exp\left(\frac{x_0 u_1}{D(p')}\right)}\right\}. \tag{3}$$

In case of Bohm diffusion $D(p) = D_0(p/m_p c)$ and one obtains:

$$f_0(p) = K \exp\left\{-\frac{3u_1}{u_1-u_2}\int_{p_{inj}}^{p}\frac{dp'}{p'}\frac{1}{1-\exp\left(-\frac{p_*}{p'}\right)}\right\}. \tag{4}$$

where $p_* = |x_0| u_1 m_p c/D_0$. Now one can show that for $p \ll p_*$, $f_0(p) \propto (p/p_*)^{-3r/(r-1)}$, with $r = u_1/u_2$, the standard result. However, for $p \gg p_*$, $f_0(p) \propto \exp\left[-\frac{3r}{r-1}\frac{p}{p_*}\right]$. The quantity $p_{max} = p_*(r-1)/3r$ plays the role of maximum momentum of the accelerated particles.

This simple example shows how a maximum momentum can be obtained in a stationary approach only by imposing the boundary condition at a finite boundary. Physically this corresponds to particles' escape, as shown by the fact that the flux of particles at $x = x_0$ is

$$\phi(x_0,p) = u_1 f(x_0,p) - D(p)\frac{\partial f(x_0)}{\partial x} = -\frac{u_1 f_0(p)}{1-\exp\left(\frac{u_1 x_0}{D(p)}\right)}\exp\left(\frac{u_1 x_0}{D(p)}\right) < 0. \tag{5}$$

The fact that $\phi(x_0, p) < 0$ shows that the flux of particles is directed towards upstream infinity. Moreover, the escape flux as a function of momentum, $\phi(x_0, p)$, is negligible for all p with the

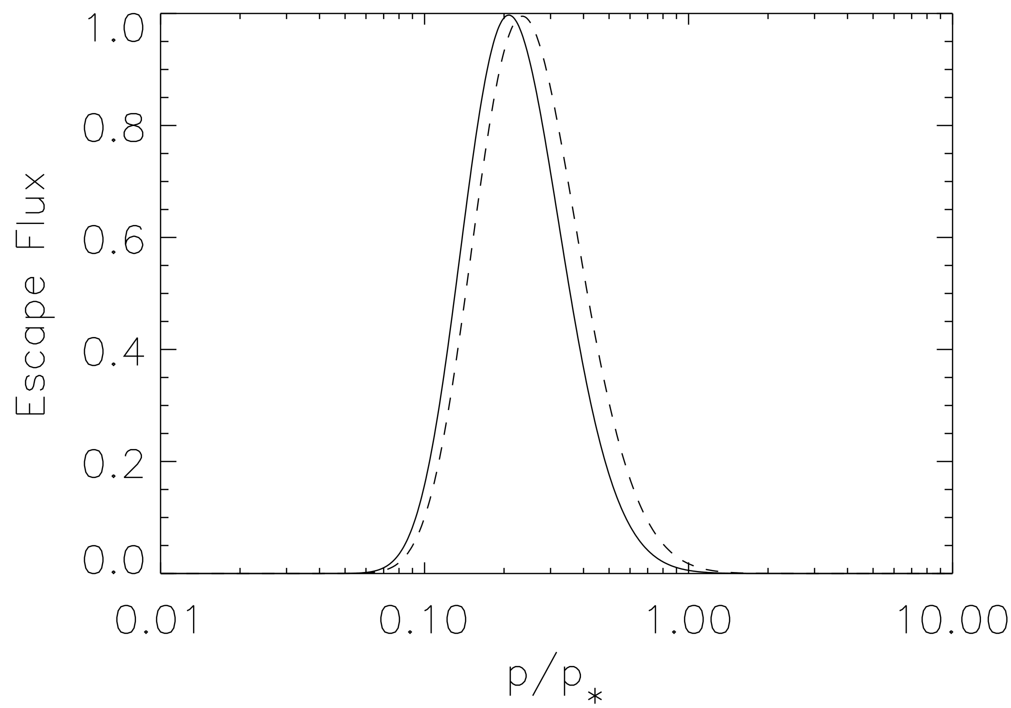

Figure 1. We plot the escape flux $\Phi(x_0, p)$ as a function of momentum. The curves refer to two different values of the shock compression ratio: $r = 4$ (solid line) and $r = 7$ (dashed line). The computation is carried out in the test-particle regime. The x-axis is in units of the reference momentum $p_* = r/(r-1)p_{max}$, while units along the y-axis are arbitrary

exception of a narrow region around p_{max}: only particles with momentum close to p_{max} can escape the system towards upstream infinity. The escape flux as a function of momentum is plotted in Fig. 1 for two values of the compression factor, $r = 4$ (solid line) and $r = 7$ (dashed line). The normalizations are arbitrary, since the calculations are carried out in the context of test particle theory. The latter value of r cannot be realized at purely gaseous shocks, but we have adopted this value to mimic the effect of shock modification, which leads to total compression factors larger than 4.

The escape phenomenon is basically irrelevant in the test-particle regime, because of the negligible fraction of energy carried by particles with $p \sim p_{max}$, but it becomes extremely important in the calculation of the shock modification induced by accelerated particles. For strongly modified shocks, the slope of the spectrum at high energies is flatter than p^{-4} and the fraction of energy that leaves the system towards upstream infinity may dominate the energy budget. This is the escape flux which appears in all approaches to cosmic ray modified shocks.

In the context of kinetic calculations of the shock modification in the stationary regime, the escape flux appears however not as a consequence of imposing a boundary condition at a finite distance upstream, but rather as an apparent violation of the equation of energy conservation

(Berezhko & Ellison (1999)), that requires the introduction of an escape term at upstream infinity. In the next section we discuss this effect, which reveals the true nature of the escape flux, as related to the form of the conservation equations and the assumption of stationarity.

3 CONSERVATION EQUATIONS AND ESCAPE FLUX

The time dependent conservation equations in the presence of accelerated particles at a shock can be written in the following form:

$$\frac{\partial\rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} \tag{6}$$

$$\frac{\partial(\rho u)}{\partial t} = -\frac{\partial}{\partial x}\left[\rho u^2 + P_g + P_c + P_W\right] \tag{7}$$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho u^2 + \frac{P_g}{\gamma_g - 1}\right] = -\frac{\partial}{\partial x}\left[\frac{1}{2}\rho u^3 + \frac{\gamma_g P_g u}{\gamma_g - 1}\right] - u\frac{\partial}{\partial x}\left[P_c + P_W\right] + \Gamma E_W \ . \tag{8}$$

Here P_g, P_c and P_W are respectively the gas pressure, the cosmic ray pressure and the pressure in the form of waves. E_W is the energy density in the form of waves and Γ is the rate at which the background plasma is heated due to the damping of waves onto the plasma. The rate of change of the gas temperature is related to ΓE_W through:

$$\frac{\partial P_g}{\partial t} + u\frac{\partial P_g}{\partial x} + \gamma_g P_g \frac{du}{dx} = (\gamma_g - 1)\Gamma E_W. \tag{9}$$

The cosmic ray pressure can be calculated from the transport equation:

$$\frac{\partial f(t,x,p)}{\partial t} + \tilde{u}(x)\frac{\partial f(t,x,p)}{\partial x} = \frac{\partial}{\partial x}\left[D(x,p)\frac{\partial f(t,x,p)}{\partial x}\right] + \frac{p}{3}\frac{\partial f(t,x,p)}{\partial p}\frac{d\tilde{u}(x)}{dx}, \tag{10}$$

where we put $\tilde{u}(x) = u(x) - v_W(x)$ and $v_W(x)$ is the wave velocity. For our purposes here we are neglecting the injection term.

Multiplying this equation by the kinetic energy $T(p) = m_p c^2(\gamma - 1)$, where γ is the Lorentz factor of a particle with momentum p, and integrating the transport equation in momentum, one has:

$$\frac{\partial E_c}{\partial t} + \frac{\partial(\tilde{u}E_c)}{\partial x} = \frac{\partial}{\partial x}\left[\bar{D}\frac{\partial E_c}{\partial x}\right] - P_c\frac{d\tilde{u}}{dx}, \tag{11}$$

where

$$E_c = \int_0^\infty dp\, 4\pi p^2\, T(p) f(p) \qquad \text{and} \qquad P_c = \int_0^\infty dp\, \frac{4\pi}{3} p^3 v(p) f(p) \tag{12}$$

are the energy density and pressure in the form of accelerated particles. Moreover we introduced the mean diffusion coefficient:

$$\bar{D}(x) = \frac{\int_0^\infty 4\pi p^2 T(p) D(p) \frac{\partial f}{\partial x}}{\int_0^\infty 4\pi p^2 T(p) \frac{\partial f}{\partial x}} \tag{13}$$

The only assumption that we made here is that $f(p) \to 0$ for $p \to \infty$.

Introducing the adiabatic index for cosmic rays γ_c as $E_c = P_c/(\gamma_c - 1)$, we can rewrite Eq. 11 as

$$\frac{\partial E_c}{\partial t} + \frac{\partial}{\partial x}\left[\frac{\gamma_c \tilde{u} P_c}{\gamma_c - 1}\right] = \frac{\partial}{\partial x}\left[\bar{D}\frac{\partial E_c}{\partial x}\right] + \tilde{u}\frac{dP_c}{dx}, \tag{14}$$

and use it to derive $u\partial P_c/\partial x$. In this way Eq. 8 becomes:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho u^2 + \frac{P_g}{\gamma_g - 1} + E_c\right] =$$
$$-\frac{\partial}{\partial x}\left[\frac{1}{2}\rho u^3 + \frac{\gamma_g P_g u}{\gamma_g - 1} + \frac{\gamma_c P_c \tilde{u}}{\gamma_c - 1}\right] + \frac{\partial}{\partial x}\left[\bar{D}(x)\frac{\partial E_c}{\partial x}\right] - v_W\frac{\partial P_c}{\partial x} - u\frac{\partial P_W}{\partial x} + \Gamma E_W. \tag{15}$$

At this point we can make use of the equation describing the evolution of the waves:

$$\frac{\partial E_W}{\partial t} + \frac{\partial F_W}{\partial x} = u\frac{\partial P_W}{\partial x} + \sigma E_W - \Gamma E_W, \tag{16}$$

where σ is the growth rate of waves, integrated over wavenumber. These quantities can be calculated once it is known how particles with given momentum p interact with waves with wavenumber k. Substituting into Eq. 15 we get:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho u^2 + \frac{P_g}{\gamma_g - 1} + E_c + E_W\right] =$$
$$-\frac{\partial}{\partial x}\left[\frac{1}{2}\rho u^3 + \frac{\gamma_g P_g u}{\gamma_g - 1} + \frac{\gamma_c P_c \tilde{u}}{\gamma_c - 1} + F_W\right] + \frac{\partial}{\partial x}\left[\bar{D}(x)\frac{\partial E_c}{\partial x}\right] - v_W\frac{\partial P_c}{\partial x} + \sigma E_W. \tag{17}$$

In the case of Alfvén waves resonant with the Larmor radius of the accelerated particles, one has $v_W = v_A = B/(4\pi\rho)^{1/2}$ and (Skilling (1975)):

$$\sigma E_W = v_A\frac{\partial P_c}{\partial x}, \tag{18}$$

so that the energy conservation equation reads

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho u^2 + \frac{P_g}{\gamma_g - 1} + E_c + E_W\right] =$$
$$-\frac{\partial}{\partial x}\left[\frac{1}{2}\rho u^3 + \frac{\gamma_g P_g u}{\gamma_g - 1} + \frac{\gamma_c P_c \tilde{u}}{\gamma_c - 1} + F_W\right] + \frac{\partial}{\partial x}\left[\bar{D}(x)\frac{\partial E_c}{\partial x}\right]. \tag{19}$$

In the general case of waves other than resonant Alfvén waves, Eq. 18 does not hold and one cannot use Eq. 19. Eq. 17 is still correct, but in order to be able to solve the problem an expression analogous to Eq. 18, relating the growth of the wave energy to the cosmic ray dynamics, is still needed.

Another point that is worth stressing is that non resonant modes, such as the ones discussed by Bell (2004), are not standard Alfvén waves (they are in fact almost purely growing modes). This causes the connection between F_W and P_W to be generally different from the standard $F_W \approx 3uP_W$, and the contribution of these waves to the energy conservation equation does not necessarily lead to Eq. 19, which was nevertheless used by Vladimirov et al. (2006).

In the following we limit ourselves to the case of Alfvén waves, which interact resonantly with particles, since in this case the calculations are all well defined.

Notice that in the stationary regime, Eq. 14, integrated around the subshock leads to

$$\frac{\gamma_c}{\gamma_c - 1}\tilde{u}P_c - \bar{D}\frac{dE_c}{dx} = \text{constant}, \tag{20}$$

because of the continuity of the cosmic ray distribution function. On the other hand, Eq. 19 (again in the stationary case) , once integrated around the shock, leads to conclude that:

$$\frac{1}{2}\rho u^3 + \frac{\gamma_g P_g u}{\gamma_g - 1} + F_W = \text{constant}. \tag{21}$$

In other words, at the subshock the energy fluxes of the gaseous and cosmic ray components are conserved separately. This is what is usually meant when we refer to the subshock as an ordinary gas shock. In the following we use the stationary version of Eq. 19:

$$\frac{\partial}{\partial x}\left[\frac{1}{2}\rho u^3 + \frac{\gamma_g P_g u}{\gamma_g - 1} + \frac{\gamma_c P_c \tilde{u}}{\gamma_c - 1} + F_W - \bar{D}(x)\frac{\partial E_c}{\partial x}\right] = 0. \tag{22}$$

4 ESCAPE FLUX AND THE NEED FOR A P_{MAX}

In non-linear theories of particle acceleration the need for a maximum momentum is dictated by the fact that the spectrum at large momenta becomes harder than p^{-4}, so that in the absence of a high p cutoff the energy content of the accelerated particle distribution would diverge. Before this happens the dynamical reaction of the accelerated particles would inhibit further acceleration. In most approaches to non-linear calculations (Malkov (1997); Malkov et al. (2000); Blasi (2002, 2004); Berezhko & Ellison (1999)), the maximum momentum is a given parameter, taken together with the assumption of stationarity of the acceleration process. The transport equation is then solved between the shock and upstream infinity. Both in the downstream region and at upstream infinity one has $D\partial f/\partial x = 0$. Moreover, at upstream infinity there are no accelerated particles ($P_c = 0$) so that Eq. 22 becomes:

$$\frac{1}{2}\rho_2 u_2^3 + \frac{\gamma_g P_{g,2} u_2}{\gamma_g - 1} + \frac{\gamma_c P_c u_2}{\gamma_c - 1} + F_W = \frac{1}{2}\rho_0 u_0^3 + \frac{\gamma_g P_{g,0} u_0}{\gamma_g - 1} \tag{23}$$

None of the calculations of particle acceleration at modified shocks carried out so far satisfies Eq. 23 unless it is *completed* with an escape flux F_{esc} such that

$$\frac{1}{2}\rho_2 u_2^3 + \frac{\gamma_g P_{g,2} u_2}{\gamma_g - 1} + \frac{\gamma_c P_c u_2}{\gamma_c - 1} + F_W = \frac{1}{2}\rho_0 u_0^3 + \frac{\gamma_g P_{g,0} u_0}{\gamma_g - 1} - F_{esc}. \tag{24}$$

Unfortunately, as showed above, this apparently harmless step is inconsistent with $f(p, x)$ being a solution of the time-independent transport equation. In fact this is not surprising since, as we stressed above, the solution of such equation cannot be characterized by a finite p_{max} when the boundary condition of vanishing $f(p, x)$ and $\partial f/\partial x$ is imposed at upstream infinity. The important conclusion that we can draw from these findings is that the problem of particle acceleration in the non-linear regime can only be described either in a time-dependent way or by assuming a boundary condition at a finite distance (e.g. Vladimirov et al. (2006)). In this second case $D\partial f/\partial x$ does not vanish at the upstream boundary and an escape flux appears in a natural way, rather than in an artificial manner as it happens in current approaches. It is therefore natural to make the following association:

$$\phi_{esc} = u_0 f(x_0, p) - D\left[\frac{\partial f}{\partial x}\right]_{x=x_0} = -D\left[\frac{\partial f}{\partial x}\right]_{x=x_0} < 0, \tag{25}$$

and the energy escape flux F_{esc} is related to ϕ_{esc} through

$$F_{esc} = \int_{p_{inj}}^{p_{max}} 4\pi\, p^2\, dp\, \phi_{esc}(p)\, T(p)\,. \tag{26}$$

In other words, no artificial escape flux needs to be introduced if a boundary condition is imposed at a finite distance upstream, or if, as an alternative, the fully time dependent solution of the problem can be found.

In next section we explore the consequences of the existence of an escape flux for the origin of cosmic rays in supernova remnants.

5 PHYSICAL MEANING OF THE ESCAPE FLUX FOR SUPERNOVA REMNANTS

The acceleration process in supernova remnants is expected to work in qualitatively different ways during the free expansion and the Sedov-Taylor phases. Here we restrict our attention to the propagation in a spatially uniform interstellar medium. During the free expansion phase the velocity of the shell remains constant and the maximum momentum grows in time in a way that depends on the growth of the turbulent magnetic field in the upstream region. During this phase particles cannot escape. Nevertheless the standard approaches to the calculation of the shock modification would lead to predict an escape flux, a symptom of the need to carry out fully time dependent calculations to treat this expansion regime. The lack of particles' escape implies an increase in the maximum momentum of the accelerated particles. This trend ends at the beginning of the Sedov-Taylor phase, when the inertia of the swept up material slows down the expanding shell.

Physically, this is the reason why we expect that the highest energies for particles accelerated in SNRs are reached at the beginning of the Sedov-Taylor phase.

During this phase, the shock velocity decreases and the magnetic field amplification upstream, as due to streaming instability, becomes less efficient. The generation of magnetic turbulence via streaming instabilities may proceed through either resonant (Bell (1978a,b)) or non-resonant (Bell (2004)) coupling between particles and waves and the two channels are likely to dominate at different times in the history of the supernova remnant (Pelletier et al. (2006); Amato & Blasi (2008)), as discussed below.

This general picture leads to a maximum momentum that decreases with time and to particles' escape towards upstream infinity: particles of momentum $p_{max}(t_1)$ do not make it back to the shock at a time $t_2 > t_1$. In other words, during the time interval between t_1 and t_2, particles with momentum between $p_{max}(t_1)$ and $p_{max}(t_2)$ escape from the system. This happens at any time, and a net flux of particles (and energy) towards upstream infinity is realized. At any given time t the spectrum of particles that escape is highly peaked around $p_{max}(t)$ (see Fig. 1 for the test-particle case). The spectrum of accelerated particles that is confined in the accelerator and advected towards downstream is cut off at a gradually lower maximum momentum, and this should reflect in the spectrum of secondary radiation, especially gamma-rays. The particles trapped downstream will also eventually escape the system, but at the time this happens they will have been affected by adiabatic losses due to the expansion of the shell. Therefore this part of the escaping flux will reflect the history of the remnant. However, it is easily seen to play a particularly important role only at the lowest energies in the cosmic ray spectrum at earth.

The flux of high energy cosmic rays, close to the knee region, as we see below, is mainly generated during the Sedov-Taylor phase and is made of particles escaping the accelerators from upstream. The actual flux of diffuse cosmic rays observed at the Earth results from the integration over time of all the instantaneous spectra of escaping particles, each peaked at $p_{max}(t)$, and from the superposition of the supernova explosions that could contribute. This integration is affected by the accelerator properties, by the dynamics of the expanding shell and by the damping processes that may affect the way the magnetic field is amplified by streaming instabilities at any given time (Ptuskin & Zirakashvili (2005)).

There is also another implication of the line of thought illustrated above: the spectrum of particles that escape, as integrated over time during the Sedov-Taylor phase, does not need to be identical to the spectrum of particles advected towards downstream. However, the latter are the particles which are responsible for the production of secondary radiation (radio, X-rays, gamma rays): the concave spectra predicted by the non-linear theory of particle acceleration and to some extent required to explain observations, might not be reflected in a concavity of the spectrum of escaping particles.

An estimate of the scalings of the relevant quantities during the ST phase can be found as follows. The radius and velocity of the expanding shell can be written as:

$$R_{sh}(t) = 2.7 \times 10^{19}\mathrm{cm} \left(\frac{E_{51}}{n_1}\right)^{1/5} t_{kyr}^{2/5} \tag{27}$$

$$V_{sh}(t) = 4.7 \times 10^{8}\mathrm{cm/s} \left(\frac{E_{51}}{n_1}\right)^{1/5} t_{kyr}^{-3/5}, \tag{28}$$

where E_{51} is the kinetic energy of the shell in the free expansion phase in units of 10^{51} erg and n_1 is the number density of the plasma upstream in units of cm^{-3}. Here we assumed the standard Sedov-Taylor time-scaling of R_{sh} and V_{sh}, but the reader should bear in mind that the adiabatic solution may be affected by the fact that in this phase the shock is radiating energy in the form of cosmic rays. The maximum energy is estimated by requiring that the diffusion length upstream equals some fraction (say 10%) of $R_{sh}(t)$. If the diffusion coefficient is assumed to be Bohm-like and the magnetic field close to the shock is $\delta B(t)$, one obtains:

$$E_{max}(t) = 3.8 \times 10^{4}\ \delta B_{\mu G}(t) \left(\frac{E_{51}}{n_1}\right)^{2/5} t_{kyr}^{-1/5}\ GeV. \tag{29}$$

The magnetic field in the shock vicinity is amplified by streaming instability, induced by the accelerated particles both resonantly and non-resonantly. Let us introduce the acceleration efficiency as a function of time: $\xi_c(t) = P_c(t)/(\rho_0 V_{sh}(t)^2)$. In terms of ξ_c, the strength of the resonantly amplified magnetic field at the saturation level can be estimated as: $\delta B^2 = 8\pi\rho_0 V^2 \xi_c / M_A$ (M_A is the Alfvén Mach number), which leads to:

$$\delta B(t) = 65\ n_1^{1/4}\ B_{0,\mu G}^{1/2} \left(\frac{E_{51}}{n_1}\right)^{1/10} t_{kyr}^{-3/10}\ \xi_c(t)^{1/2}\ \mu G\ . \tag{30}$$

In a similar way, the strength of the field in the case of non-resonant amplification can be estimated from $\delta B^2 = 2\pi\rho_0 (V_{sh}(t)^3/c)\xi_c(t)$ and leads to:

$$\delta B(t) = 198\ n_1^{1/2} \left(\frac{E_{51}}{n_1}\right)^{3/10} t_{kyr}^{-9/10}\ \xi_c(t)^{1/2}\ \mu G\ . \tag{31}$$

In general the two channels of magnetic field amplification work together but the non-resonant channel dominates at earlier times and leads to stronger magnetic field amplification.

The maximum momentum in the two cases is as follows:

$$E_{max}(t) = 2.5 \times 10^{6} \left(\frac{E_{51}}{n_1}\right)^{1/2} n_1^{1/4} B_{0,\mu G}^{1/2}\ \xi_c(t)^{1/2}\ t_{kyr}^{-1/2}\ GeV, \tag{32}$$

in the resonant case, and

$$E_{max}(t) = 7.3 \times 10^6 \left(\frac{E_{51}}{n_1}\right)^{7/10} n_1^{1/2}\, \xi_c(t)^{1/2}\, t_{kyr}^{-11/10}\, GeV \tag{33}$$

in the non-resonant regime.

In the naive assumption that the acceleration efficiency is constant in time, we see that $E_{max}(t)$ scales with time as $t^{-11/10}$ at earlier times and as $t^{-1/2}$ at later times, when resonant scattering dominates. In actuality the scalings will be more complex because of the non-linear effects (especially the formation of a precursor upstream) induced by accelerated particles, which also lead to a time dependence of $\xi_c(t)$.

As discussed in the previous sections, it is not clear how to describe the non-resonant waves in the context of the conservation equations. A calculation of the dynamical effect of these modes on the shock is therefore not reliable at the present time. For this reason, here we confine ourselves to the investigation of the effects of resonant waves, for which there is no ambiguity. It is however worth keeping in mind that the introduction of the non-resonant waves is likely to result in significantly higher maximum energies at the early stages of the Sedov-Taylor phase.

Our complete calculations, including the non-linear dynamical reaction of the accelerated particles, the resonant amplification of magnetic field and the dynamical reaction of the field itself have been carried out as discussed by Caprioli et al. (2008b). The results are illustrated in Figs. 2-4. In the left panels we plot the maximum momentum (p_{max}), the shock velocity (V) and the two compression factors (R_{sub} and R_{tot}) as functions of time. The right panels show the acceleration efficiency and the escape flux normalized to $\rho_0 V_{sh}^2$ and $(1/2)\rho_0 V_{sh}^3$ respectively, and the strength of the downstream magnetic field in units of $10^3 \mu G$. The maximum momentum and the shock speed are in units of $10^6 m_p c$ and $10^8 \mathrm{cm\,s^{-1}}$ respectively. The three figures refer to the following sets of parameters: $n_0 = 0.1 \mathrm{cm}^{-3}$, $B_0 = 1\mu G$ (Fig. 2), $n_0 = 0.1 \mathrm{cm}^{-3}$, $B_0 = 5\mu G$ (Fig. 3) and $n_0 = 0.03 \mathrm{cm}^{-3}$, $B_0 = 1\mu G$ (Fig. 4).

The maximum momentum is determined at each time by requiring that the diffusion length in the upstream section equals $0.1 R_{sh}(t)$. The quantities $p_{max}(t)$, $R_{sub}(t)$ and $R_{tot}(t)$ are all outputs of the non-linear calculations at the time t. The first point in time in all figures corresponds to the beginning of the Sedov phase. The time at which the Sedov-Taylor expansion begins was determined assuming $E_{51} = 1$ and that the mass of the ejecta is $M_{ej} = 5 M_\odot$. The other relevant parameters are the temperature of the ISM in which the SNR is expanding, for which we assumed $T_0 = 10^4\ K$, and the momentum threshold for particles to be injected into the accelerator, which was chosen to be $p_{inj} = \xi_{inj} \sqrt{2 k_B T_2 m_p}$, with $\xi_{inj} = 3.8$ and T_2 the temperature downstream of the shock (see Caprioli et al. (2008b) for details).

Some general comments are in order: one may notice that the total compression factors obtained in our calculations are always lower than $\sim$ 10. This is uniquely due to the dynamical reaction of the amplified magnetic field. As shown by Caprioli et al. (2008a) the effect of the amplified field on the plasma compressibility is relevant whenever the magnetic pressure becomes comparable with the thermal pressure of the background plasma upstream. The consequent decrease in the compression ratios allows us to be consistent with the values that have been inferred from observations of a few SNRs (Warren et al. (2005)).

The highest momentum of accelerated particles, as expected, is reached at the beginning of the Sedov-Taylor phase and is of order $\sim 10^6$ GeV (about the knee). This should be considered as a lower limit to the maximum energy reached at that time, since we have decided not to include the non-resonant channel of magnetic field amplification, which is very efficient when the shock velocity is large. During the following expansion, the time-dependence of p_{max} is reasonably well approximated by $p_{max} \propto t^{-1/2}$, in agreement with Eq. 32, since $\xi_c(t)$ is roughly constant (see solid curve on the right panels of Figs. 2-3-4 and discussion below).

A crucial ingredient in calculating the maximum energy at a given time is the strength of the magnetic field. The magnetic field intensity in the downstream plasma is plotted in the right panels (dot-dashed line) for the three cases considered here. The typical values are between $\sim$ a few$-10\mu G$ at late times and $\sim 30 - 100\mu G$ at the beginning of the Sedov phase. After the first few thousand years, the dependence on time is not far from $\delta B_2 \propto t^{-3/10}$, as would result from Eq. 30, using the additional information on the approximate constancy of $\xi_c(t)$ and R_{sub} (dashed line in the left panels of Figs. 2 to 4). This scaling of B_2 with time is approximate and not obvious to expect. In fact, the situation to which the plot refers is more complicated than that described by Eq. 30, where a number of effects have been neglected, first among these the presence of a precursor which evolves with time (R_{tot} is changing as can be seen from the dot-dashed curve on the left of Figs. 2 to 4) and the time varying adiabatic compression it entails.

The right panels also show the acceleration efficiency (solid line) and normalized escape flux (dashed lines). One should notice that even when the acceleration efficiency is very high, of order $\sim 50 - 60\%$, the escape energy flux never exceeds $\sim 30\%$. As discussed above, this latter quantity should be the one that is more directly related to the cosmic ray energetics in the Galaxy, at least at the highest energies, while the former is more relevant for the generation of secondary radiation due to cosmic ray interactions in the acceleration region.

The acceleration efficiency and the normalized escape flux initially increase with time during the Sedov-Taylor expansion phase. This behaviour is related to an analogous trend of the shock

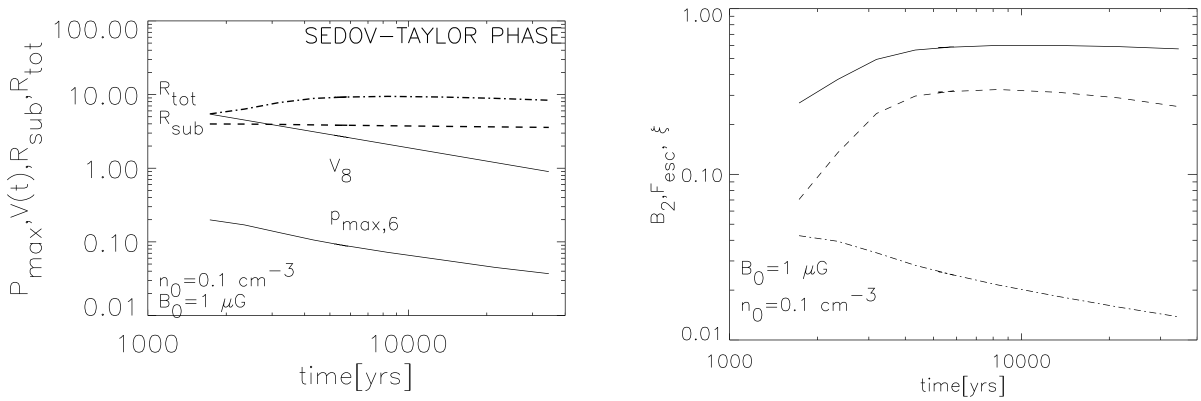

Figure 2. Left panel: time dependence of the maximum momentum of accelerated particles (solid curve labeled as $P_{max,6}$) in units of $10^6 m_p c$, of the shock velocity in units of 10^8 cm/s (solid curve labeled as V_8), of the compression factor at the subshock R_{sub} (dashed curve) and of the total compression factor R_{tot} (dot-dashed curve), during the Sedov-Taylor phase. Right panel: time dependence of the magnetic field strength downstream of the subshock in units of $10^3\ \mu G$ (dot-dashed curve), of the escape flux normalized to $\rho_0 V_{sh}^3/2$ (dashed curve) and of the cosmic ray pressure normalized to $\rho_0 V_{sh}^2$. The magnetic field strength, B_0, and the number density of the background plasma, n_0, at upstream infinity are taken to be $B_0 = 1\ \mu G$ and $n_0 = 0.1$ cm^{-3}.

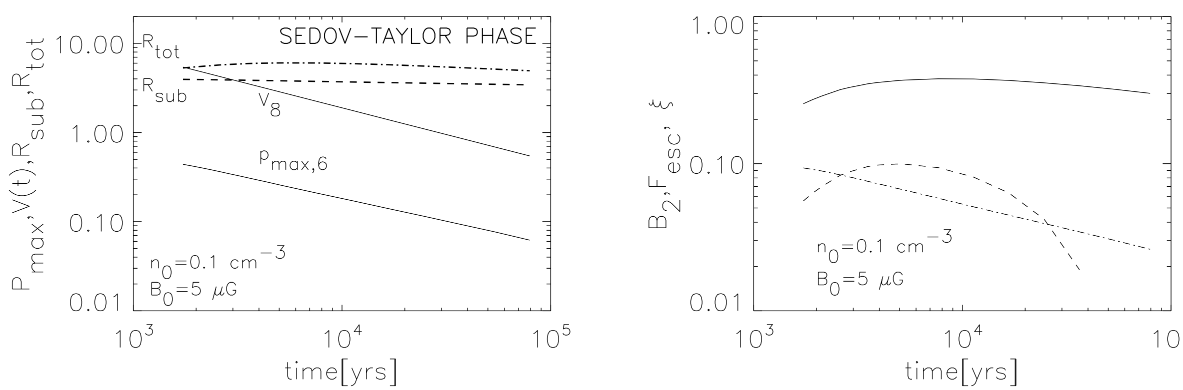

Figure 3. Same as Fig. 2 but for $B_0 = 5\ \mu G$ and $n_0 = 0.1$ cm^{-3}.

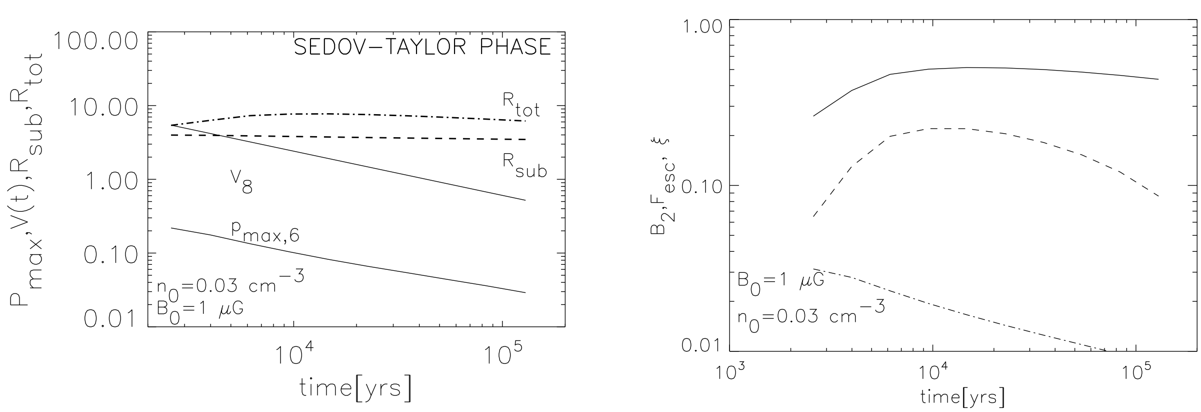

Figure 4. Same as Fig. 2 and Fig. 3 but for $B_0 = 1\ \mu G$ and $n_0 = 0.03$ cm^{-3}.

modification, as can be clearly seen from the time-dependence of R_{tot} (dash-dotted curve in the left panel of Figs. 2-4). In fact, at the beginning of the Sedov phase, the amplified magnetic field is at a maximum and its dynamical reaction on the shock is so strong that injection gets suppressed and the acceleration efficiency is reduced. As soon as the magnetic field strength starts decreasing, the shock modification increases, and ξ_c and F_{esc} with it. Notice, however, that this does not mean that the actual cosmic ray pressure and escape flux increase, because ξ_c and F_{esc} are normalized to $\rho_0 V_{sh}^2(t)$ and $\rho_0 V_{sh}^3(t)/2$ respectively, and both decrease with time rather quickly.

At later times, both ξ_c and F_{esc} start decreasing, with the latter showing a more rapid decline than the former. This is due to the fact that the shock is slowing down and progressively becoming unmodified: the maximum momentum is decreasing and the spectrum of accelerated particles is steepening. Recalling again that the plots show normalized quantities, one gathers that the decline with time of cosmic ray pressure and escaping energy flux is quite dramatic in this phase.

6 CONCLUSIONS

From the physical point of view, escape of particles towards upstream of a shock is the only way (in the absence of energy losses) to reach some sort of stationarity. This flux can be calculated, even in the test-particle case, by assuming that the distribution function vanishes at some finite distance upstream of the shock. This is simply a mathematical way to describe the fact that at some distance particles freely stream away. Their density does not actually vanish at that point but is substantially reduced compared with the case when diffusion enhances the local density of particles. In a time dependent calculation with the boundary condition at upstream infinity, the escape should translate in the fact that at some point the time needed for particles with sufficiently high momentum to return to the shock exceeds the age of the shock itself.

Independent of the way in which one pictures this phenomenon, the escape flux is a fact. In test-particle approaches to shock acceleration, it is not a very important fact, because of the negligible amount of energy carried by the tail of the particle distribution. However, in the case of non-linear particle acceleration the escape of particles from the shock is inherently important: it is required by energy conservation and indeed allows to actually reach some sort of equilibrium; morevoer, it makes the shock, in a sense, radiative, which implies larger compression of the plasma and therefore larger shock modification.

The escape flux has long been known to be a characteristic feature of all stationary approaches to non-linear shock acceleration (see e.g. Berezhko & Ellison (1999) and Blasi et al. (2005) among others). A key point for its interpretation is the fact that it appears even in situations in which the escape of particles upstream of the shock is physically impossible. This is the case, for instance, of the free expansion phase of SNR shells, where an energetically important escape flux is found despite the fact that the maximum momentum of the accelerated particles is increasing with time.

We conclude that the appearance of the escape flux is related to the requests of stationarity and at the same of the existence of a finite maximum momentum. All treatments of non-linear shock acceleration are bound to assume a finite maximum momentum: the flatness of the spectrum at high energies would otherwise cause the energy contained in the cosmic ray distribution to diverge. However, a finite p_{max} is not consistent with an infinite size of the acceleration region, under the assumption of stationarity and in the absence of losses. The escape flux is there to highlight this problem and we proved this by using the conservation equations in their most complete form.

Besides proposing a physical interpretation of the escape flux, we investigated its behaviour during the Sedov-Taylor phase of a supernova remnant, likely the most relevant context for the production of the diffuse cosmic ray spectrum observed at the Earth. We showed that the escape flux may involve between few and 10-30 % of the shock ram pressure, while the particle acceleration efficiency at the same time reaches 50-60 %.

The maximum energy up to which particles may get accelerated is reached at the beginning of the Sedov phase and is of order 10^{15} eV if only resonant amplification of the field is included. E_{max} might be larger for some SNRs that at the very beginning of the Sedov-Taylor phase may experience the effect of non-resonant streaming instability (Bell 2004). This mechanism provides extremely efficient field amplification as long as the shock velocity is high, and hence is expected to play a very important role in the early Sedov-Taylor phase (and possibly during the free expansion phase). At present, non-resonant modes cannot be formally accounted for in the conservation equations, since a detailed description of the energy transfer between particles and waves is not available yet. We carried out all calculations in the simpler case of Alfvén waves interacting with particles in a resonant way.

The amplification of the magnetic field, by either resonant or non-resonant streaming instability, has profound implications on the escape flux of particles towards upstream of a shock, and therefore on the spectrum of cosmic rays we observe at the Earth. The most obvious consequence of the magnetic field amplification is that of allowing for higher values of the maximum energy of accelerated particles, as shown by our Eqs. 32 and 33. However large magnetic fields exert a dynamical reaction on the plasma leading to a reduction of the compression in the precursor. This happens whenever the magnetic pressure exceeds the pressure of the background gas (Caprioli et al. 2008a). As a result, the concavity of the spectrum of accelerated particles (Caprioli et al. 2008b) is reduced and at the same time the escape flux at $p \sim p_{max}$ decreases. It follows that larger field implies larger p_{max} but not necessarily larger escape flux, as shown in Figs. 4, 2 and 3 (see the behaviour of the curves at early times).

The escape of accelerated particles from a cosmic ray modified shock has profound implications for the origin of cosmic rays, which will be discussed in detail in a forthcoming paper. Here we want to emphasize some general points:

1) the escape from upstream is the natural solution to the well known problem of explaining how the highest energy particles (say, close to the knee energy) could escape the system without suffering substantial adiabatic energy losses;

2) the magnetic field amplification is expected to switch from mainly non-resonant to mainly resonant at the beginning of the Sedov-Taylor phase. It can be easily understood that this may lead to peculiar changes in the spectrum of cosmic rays detected at the Earth, reflecting this transition;

3) the flux of escaping particles, once integrated in time during the SNR evolution may be very different from the concave instantaneous spectrum which can potentially be observed in a SNR, for instance by looking at its multifrequency emission. This point is certainly relevant for the purpose of addressing the commonly asked question of how a concave spectrum of accelerated particles can reflect in an almost perfect power law over many orders of magnitude;

4) there is a further complication of all the picture, due to the acceleration of nuclei at energies that may be expected to scale as the charge of the nucleus (in the case of Bohm diffusion). Any calculation of the flux of single chemical species observed at the Earth must take these complex effects into account.

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